Exercise Sheet 7 due 4 December 2014

1. *Hermitian matrices*

   i. Show, for Hermitian operators $\hat{A}$ and $\hat{B}$, that the product $\hat{A}\hat{B}$ is a Hermitian if and only if $\hat{A}$ and $\hat{B}$ commute.

   ii. Prove that the operator that is the commutator $[\hat{A}, \hat{B}]$ of two Hermitian operators $\hat{A}$ and $\hat{B}$ is never Hermitian, unless it is zero. Do you see a way for making the non-vanishing commutator Hermitian?

2. *Uncertainty relation*

Consider a mass of 1 μg, whose position we know to a precision of 1 μm.

   i. What would be the minimum uncertainty in its velocity in a given direction?

   ii. What would be the corresponding uncertainty in velocity if the particle was an electron instead?

3. *Pauli matrices*

The Pauli matrices are defined as

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

   i. Calculate $\vec{\sigma}^2 = \hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_z^2$.

   ii. Find the eigenvalues and (normalized) eigenvectors $|\chi_{z,n}\rangle$ of $\hat{\sigma}_z$.

   iii. Find the eigenvalues and (normalized) eigenvectors $|\chi_{x,n}\rangle$ of $\hat{\sigma}_x$.

   iv. Show by explicit calculation that $\sum_n |\chi_{x,n}\rangle\langle\chi_{x,n}|$ is the identity matrix.

   v. Determine the commutators between each pair of the Pauli matrices by explicit matrix multiplication. Simplify your answer as much as possible and compare to the Pauli matrices.

   vi. Calculate $\exp(\sigma_x)$ by transforming to the eigenbasis and, alternatively, by using the power series.